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Interesting Discoveries Through Technologies and Reflections Along Circles

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Abstract

This paper is an expansion of [11]. The discussions in this paper were inspired by a college entrance practice exam from China. The investigations lead us to apply technological tools to explore the reflections along circles. Firstly, we shall see if we choose the incoming and outing light beams to be at a specific angle within a circle, we shall create many nice geometric patterns. Secondly, we prove that the reflections along the circle are periodic if the reflection angle is a rational degree. Thirdly, we replace the straight lines for light beams by two symmetric curves with respect to the corresponding normal line at the point on the circle, we will create nice patterns involving curves. Finally, we use technological tools to explore interesting scenarios when reflections are done along ellipses.

1 Introduction

In this paper, we use technological tools to explore and investigate reflections of light beams or billiards along a smooth curve. The problems discussed in this paper were inspired by a college entrance practice problem (see Example 1 in Section 2) we found from China (see [5]). In short, if we start from a point and starts bouncing against the curve (like a straight line light beam), we would like to know when the bounces will come back to the initial starting point. In such cases, we call the reflections to be periodic. One can view the incoming rays (incidental rays) and the outgoing rays (reflected rays) at any point of the curve as inverses with respective to the normal line at the point on the curve. In Section 3, we discuss scenarios that will produce many nice geometric patterns when we choose a *proper angle* between the incoming ray and the normal vector at a point on the boundary of a circle, which we call it the reflection angle. Inspired by [4], where the reflections along ellipses are periodic sometimes, we prove that the reflections are periodic for circles in Section 4 when the reflection angles are rational degrees. In Section 5, we replace the incoming and outgoing rays from lines to curves, based on the formula 9 derived from [10]. As a result, we may create beautiful patterns involving curves. To encourage beginners to appreciate how technological tools can inspire learning interesting mathematics, in Section 6, we use technological tools to explore three known facts about reflections along ellipses.

2 A College Entrance Practice Problem From China

We present the following Example, which is originated from a college entrance practice problem from China, see [5].

Example 1 *We refer to the following figure: A light beam starts from $M(x_0, 4)$ and follows the direction parallel to the $x - axis$ and hits $y^2 = 8x$ at P and reflects and touches the horizontal parabola at Q then the light beam touches the line $x - y - 10 = 0$ at the point N. Find x_0 if the final reflection at N comes back to M.*

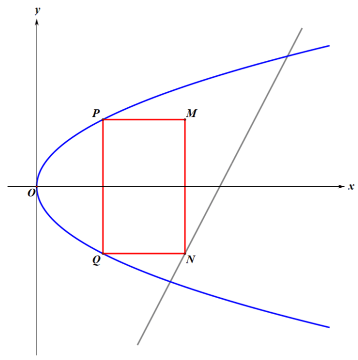

Figure 1. Reflections between a horizontal parabola and a line

First we note that the slope of the tangent line at a point on $y^2 = 8x$ (or $y^2 - 8x = 0$) satisfies $2y\frac{dy}{dx} - 8 = 0$, which implies $\frac{dy}{dx} = \frac{4}{y}$. Since $P = (x, 4)$ lies on $y^2 - 8x = 0$, we see $x = 2$. We also note that $\frac{dy}{dx} = 1$ at P, thus the angle between MP (parallel to the $x - axis$) and the tangent line at P is $\frac{\pi}{4}$. Since the normal vector n_P is perpendicular to the tangent vector at P, the incidental angle $\theta = \frac{\pi}{4}$ too. Thanks to the law of reflection, we see $PQ \perp MP$ and $\frac{dy}{dx}$ at $Q = (2, -4)$ is -1. Analogously, we see $QN \perp NM$. We plug $y = -4$ into the line of $x - y - 10 = 0$ yields $x_0 = 6$.

We recall that if a straight line in the plane has the form of $ax+by+c=0$ and if $(u,v) \in \mathbb{R}^2$, then the reflected point (u', v') of (u, v) with respect to the line $ax+by+c=0$ will have the form of

$$\begin{aligned} u' &= u - \frac{2a\,(au+bv+c)}{a^2+b^2}, \\ v' &= v - \frac{2b\,(au+bv+c)}{a^2+b^2}. \end{aligned} \tag{1}$$

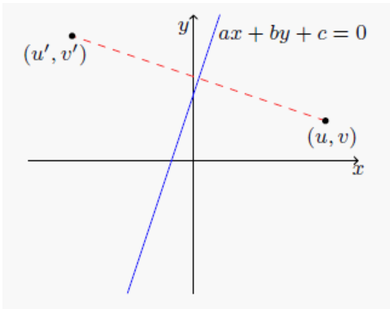

Figure 2. Reflection of a point with respect to a line

Alternatively, we may also call (u', v') to be the inverse of (u, v) with respect to the line $ax + by + c = 0$. We recall a game called 'Brick Breaker Arcade', which demonstrates simple applications on light or billiards reflections. Readers can recall some fun from the following videos, see [1] or [2]. Now we consider the following scenario, which we can develop as a game similar to 'Brick Breaker Arcade'.

Example 2 *Given a circle of $x^2 + y^2 = 4$ and a point $A = (0.385, 0.805)$ in the interior of the curve. We start with the initial incidental ray of $\overrightarrow{AB} = (0.705\,86, 0.871\,32)$, where $B = (1.09086, 1.67632)$ is a point on the circle. Select the proper point D and the line L so that the second reflection followed by L will come back to the starting point A after two reflections along the circle.*

Remark: A game easily linked to this problem can be stated as follows: We start with a point A within a given circle and start a random reflection along the circle, and we will be looking for a precise place within the circle (point D) and a proper line (L) so that the reflection will come back to the point A after finitely many reflections.

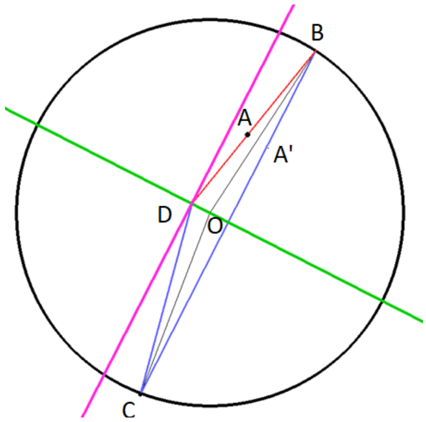

Figure 3. Billiard reflections within a circle and a secant line

We outline how we approach Example 2 as follows:

Step 1. The line normal to the circle O at $B = (x_1, y_1)$ in rectangular form is $OB : y = 1.5367x$,

Step 2. We find the reflection A with respect to OB, which we call it A'. The line equation $A'B$ is $y = 1.954x - 0.4557$. Next we find the point of intersection between $A'B$ and the circle to be $C = (-0.7212729048, -1.865412929)$.

Step 3. We note the normal line at C is $y = 2.5863x$. Next we find the reflection of BC with respective to the line OC to be $y = 3.685x + 0.7926$, which we call it L'. In other words, we can view the incoming rays, BC, and the outgoing ray L' at C of the circle, as inverses with respective to the normal line at the point C on the circle.

Step 4. We find the intersection point between L' and AB to be $D = (-0.188\,84, 0.096\,64\,7)$.

Step 5. It suffices to find the line of angle bisector, L'', between DB and CD at the point D. This turns out to be $y = -0.5117x$, which is the green line in Figure 3.

Step 6. Finally, we find the desired line L''' (shown in pink in Figure 3), which is perpendicular to L'' and passes through the point D, to be $y = 1.954x + 0.4657$.

2.1 Explorations

To design an exam type of question, it is understandable that the problem cannot be too complicated and the answer has to be simple too. However, we may make a problem more realistic if technological tools are available to students. For example, one can explore the following scenarios:

1. Repeat Example 2 by choosing a different point A in the interior of the circle and different boundary point B on the circle. To experiment this interactively with Chinese Netpad [8], click on [S1].

2. Repeat Example 2 by starting a proper interior point A and the boundary point B. Now we want to find the proper point D and the line L so that the sixth reflection followed by L will come back to the starting point A. To experiment this interactively with Chinese Netpad [8], click on [S2].

3. Given point $M(x_0, y_0)$ is fixed. Now we make the line $ax + by + c = 0$ to be movable. Move the individual variable a, b or c, so that the final reflection comes back to the point M? To experiment this interactively with Chinese Netpad [8], click on [S3].

4. Suppose the given point $M(x_0, y_0)$ and the line of $ax + by + c = 0$ are fixed. Make the point P on the curve $y^2 = 8x$ be movable, after the reflections of MP, PQ and QN, when will the last reflection NM come back to the point M? To experiment this interactively with GeoGebra [6], click on [S4].

3 Geometric Patterns, Reflections and Circles

We now turn to a natural question one would ask by connecting an interior point of a given circle with another point that lies on the boundary of the circle. The question we ask is if such initial starting ray will come back to the same starting point after finitely many reflections. In other words, we ask if the reflections will become periodic after finitely many steps. We exclude the trivial case where the first incoming light beam is the normal vector to a given point. At present we focus on the case when the simple closed curve is a circle. We pick the starting point from an interior point E of a circle with the trajectory that hits a point P_1 on the boundary of the circle. At the point P_1, we define the angle θ of incidence as the angle between the inward pointing normal vector at point P_1 and the billiard trajectory EP_1. Similarly, define the angle of reflection (or simply reflection angle) as the angle ϕ between the normal vector at P_1 and the billiard trajectory P_2P_3. We see the angle of incidence θ is same as the angle ϕ of reflection (See Figure 4). Since $\theta = \phi$, we simply call such angle as reflection angle in many places in our paper with no confusion. We first analyze the angle θ when the reflections form a regular polygon. With a dynamic geometry software (DGS) at hand, we start with a point $E \in \mathbb{R}^2$ with a fixed direction v, which forms a fixed angle θ with the normal vector at P_1 on the circle. We continue with the reflection with the fixed angle θ and ask if there is a positive integer n so that the $P_n = E$. If such positive integer n exists, we call such reflection to be periodic.

Before proving the reflections along a circle is periodic when the incidental angle is rational, we incorporate both dynamic geometry system (DGS), such as GeoGebra [6] and a computer algebra system (CAS) such as Maple [7] to *experiment* and *conjecture* if the minimum number of needed reflections for making the reflections periodic can be found. We use examples mentioned in Section 3.1 as demonstrations. Accordingly, we need to specify in advance how two points can be numerically considered as the same point. For example, we may set a pre-determined numerical small tolerance to be $\epsilon > 0$, and for the points $p = (x_1, y_1)$ and $q = (x_2, y_2) \in \mathbb{R}^2$ satisfying $\|p - q\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} < \epsilon$, we say p and q are identical within ϵ. We later prove that the circle reflections are periodic if the reflection angle is a rational degree in Theorem 8, and the minimum number of reflections can be expressed explicitly once the reflection angle is given in Corollary 9. In such case, the tolerance $\epsilon > 0$ will no longer be needed.

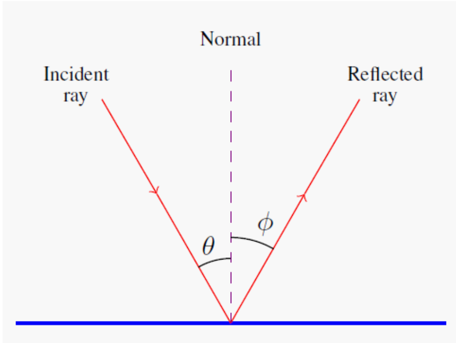

Figure 4. Law of reflection

3.1 Explorations On Reflections Along Circles

Since we ask if a reflection becomes periodic or not, it is natural we first consider the case of a regular *convex polygon* that is inscribed in a circle. We recall that *a convex polygon is a simple polygon (not self-intersecting) in which no line segment between two points on the boundary ever goes outside the polygon. A convex polygon is regular if each side is of equal length*; subsequently, each interior angle of a regular convex n-polygon has the measurement of $\left(1-\frac{2}{n}\right)\cdot 180° = \left(1-\frac{2}{n}\right)\pi$. Therefore if the incidental angle for a reflection, or simply reflection angle, is

$$\theta = 90°\left(1-\frac{2}{n}\right) = 90°\left(\frac{n-2}{n}\right) = \frac{\pi}{2}\left(\frac{n-2}{n}\right), \tag{2}$$

where $n = 3, 4, \ldots$. Then the reflections become periodic and follow the path of a regular convex n-polygon.

For example, when $n = 3$ in (2) we see the reflection angle $\theta = 30°$, then we create an equilateral. In the following Figure 5, we consider the circle $x^2+y^2 = 4$ and start with the initial incoming ray of EA, with the interior point $E = (0.276886, 1.09285)$ and $A = (1.45596, 1.3712)$, which lies on the circle. We see the inclination angle θ between EA and the normal line at A is $\theta = 30°$. *It follows that the third reflection at the point C, will come back to the initial starting point E. We call the number of reflections, which make the reflections periodic, to be* 3. In the meantime, we see the reflections form an equilateral triangle.

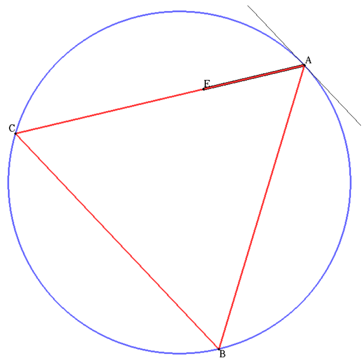

Figure 5. Reflections and an equilateral.

We note the following observations:

1. The reflections become periodic or not does not depend on the location or the size of the circle.
2. If we assume the initial ray starts with a point $E \in \mathbb{R}^2$ and ends with the point $P_1 = (a, 0)$ on the circle of $x^2 + y^2 = a^2$. Then the incidental angle $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. For simplicity, we assume $\theta \in \left(0, \frac{\pi}{2}\right)$ in our discussions.

Next, with technological tools, one can conjecture that there are other scenarios that will make the reflections along a circle periodic. For example, we now consider *a regular star polygon, that is a self-intersecting, equilateral equiangular polygon.*

Example 3 *Consider the incidental angle of $\theta = 15°$ with $E = (0, e)$ and $P_1 = (a, 0)$ on the circle of $x^2 + y^2 = a^2$. We show that such reflections produce a regular (star) 12-polygons. Incidentally, we produce the regular (convex) 12-polygon by connecting adjacent points on the circle and also another inner regular (convex) 12-polygons.*

Here, we use $e = a \tan \theta$ to find E. If $a = 2$, then $E = (0, 0.5358983848)$. We depict the produced regular (star) 12-polygons using [6] and [7] in the following Figures 6(a) and 6(b) respectively. To experiment this interactively with GeoGebra [6], Maple [7] and Netpad [8] we refer readers to [S5], [S6] and [S7] respectively.

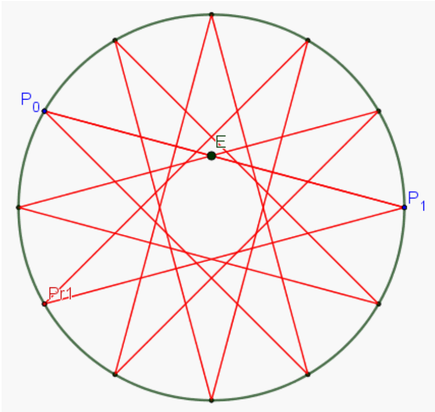

Figure 6(a) When $\theta = 15°$ and [6]

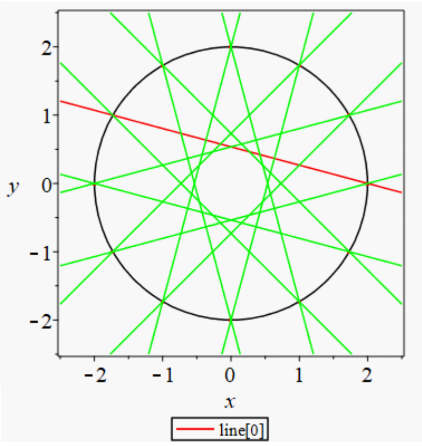

Figure 6(b) When $\theta = 15°$ and [7]

Now we turn to an interesting curve that if found after reflections along a circle as follows:

Definition 4 *We call caustic curve to be the curve such that each billiard trajectory is tangent to such a curve.*

We see the caustic curve when $\theta = 15°$ as we see in Figures 6(a) or 6(b) is a regular convex 12-gons. We use technological tools to experiment another example as follows:

Example 5 *Consider the incidental angle of* $\theta = 5°$ *and* $P_1 = (2, 0)$, *then we obtain a regular star* 36*-polygons and* $E = (0, 0.1749773271)$ *in this case. We depict the regular* 36*-polygons using GeoGebra [6] and Maple [7] respectively in Figures 7(a) and 7(b) respectively. We remark that the caustic curve in this case is a convex regular* 36*-polygons.*

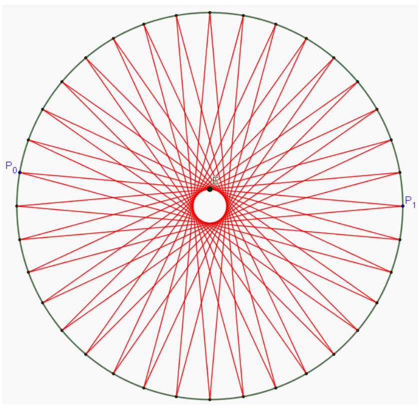

Figure 7(a) When $\theta = 5°$ *and [6]*

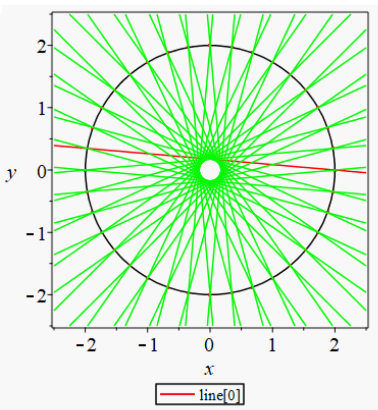

Figure 7(b) When $\theta = 5°$ *and [7]*

At this point we rely on a CAS such as Maple ([7]) to conjecture the relationship between the reflection angle θ and the number of regular polygon it may create if the reflections become periodic. We consider the following Examples.

Example 6 *We set $\theta = \frac{10\pi}{180}$ or 10° to be the reflection angle, we want to find the point E and the number points n needed to make the refection periodic.*

With the help of Maple, we get a regular star 9- polygon. In this case, $E = (0, 0.3526539614)$, which shows the initial and final reflections as follows in Figure 9(a) and 9(b).

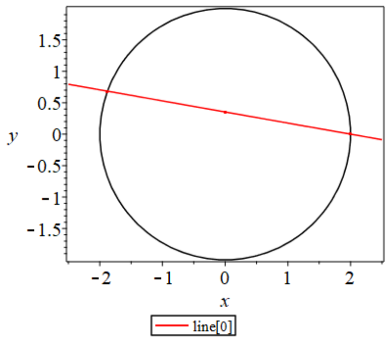

Figure 9(a) Iniitial ray of EP_1 when $\theta = 10^\circ$ with [7]

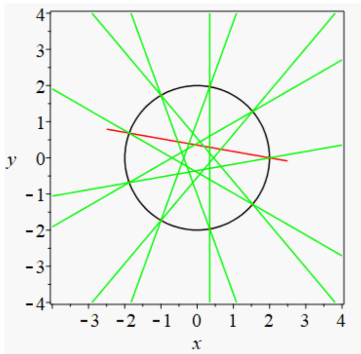

Figure 9(b) When $\theta = 10^\circ$ and [7]

Example 7 *If we start with $E = (0, e)$, the point $P_1 = (2, 0)$. If we set $\theta = \frac{\pi}{180}$ or 1°. Then (a) find the number of points needed to make the reflections periodic; (b) find the position of E to see the reflections periodic.*

We may use Maple to compute the number points n needed to make the refection recursive, we get a regular star 180−polygons. In this case, $E = (0, 2\tan\theta) = (0, 0.03491012986)$, which show the initial and final reflections as follows in Figure 10(a) and Figure 10(b) respectively. We shall see from Theorem 17 that the caustic curve in this case is a tiny regular convex 180−polygon, that circumscribes the circle centered at $(0, 0)$ with radius $\sin 1^\circ$, which is difficult to visualize in this case.

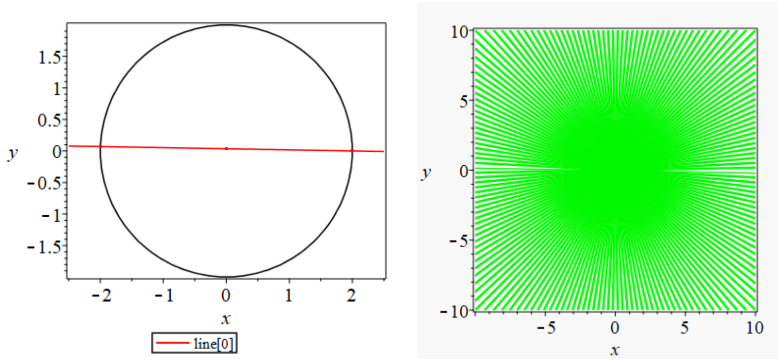

Figure 10(a) Initial ray when $\theta = 1°$ Figure 10(b) Final reflection when $\theta = 1°$

3.2 Reflections are periodic when the reflection angle is rational

Now we proceed to prove that the reflections along the circle shall become periodic if the reflection angle is rational.

Theorem 8 *Let the starting point S be in the interior of circle C of $x^2 + y^2 = 1$ (see Figure 11). Let A_0 be on the circle and θ represent the reflection angle of rational degrees $\frac{q}{p}$, which is the angle between SA_0 and the normal vector at A_0. Then there exists positive integers m and and n such that the reflections are closed after n times of reflections and m times of rotation for A_0 with respective to the origin. If in addition, $\gcd(m, n) = 1$, then n represents the smallest positive number for the reflections become periodic.*

Proof: We note that each point of the reflection on the circle travels the arc length of $\pi - 2\theta$. We are looking for the least positive integer m satisfying the equation of

$$n(\pi - 2\theta) = 2m\pi$$

for some positive integer m. We note that above equation is equivalent to finding two positive integers, m and n satisfying

$$\frac{\pi - 2\theta}{2\pi} = \frac{m}{n},$$

which means that after n times of reflections with angle θ, returning to A_0, we have also gone through m times of rotation for A_0 with respect to the origin O. Now suppose $\gcd(m, n) = 1$, and if there exists another positive integer $n' < n$ for which the reflections become periodic, then there exists another positive integer m' satisfying $n'(\pi - 2\theta) = 2m'\pi$, which implies that

$$\frac{m}{n} = \frac{m'}{n'}.$$

In other words, we have $mn' = nm'$, which implies that mn' is divisible by n. Since $\gcd(m, n) = 1$, we see that n' is divisible and $n' \geq n$, which is a contradiction.

Corollary 9 *If the inclination angle $\theta = \frac{q}{p}$ in degree(s), and $d = \gcd(90p - q, 180p)$, then $n = \frac{180p}{d}$ is the minimum positive integer for making the reflections periodic and $m = \frac{90p-q}{d}$ is the number of rotation for the point A_0 with respective to the origin O.*

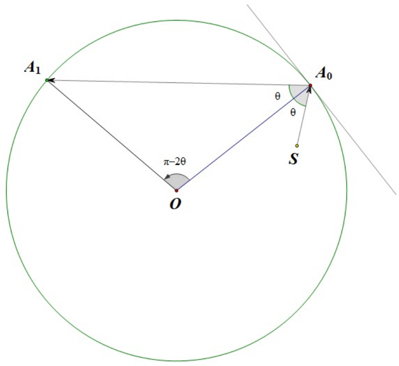

Figure 11. Reflections over a circle

Proof: Suppose $\theta = \frac{q}{p}$. The theorem follows immediately from the following observations.

$$\begin{aligned} n\left(\pi - \frac{\frac{2q}{p}\pi}{180}\right) &= 2m\pi, \\ \frac{n\pi}{90p}(90p - q) &= 2m\pi, \\ \frac{90p - q}{180p} &= \frac{m}{n}. \end{aligned} \tag{3}$$

4 Discussions

Obviously, when $\theta \in (0, 90)$, we see $n > 2m$. In other words, the minimum reflection number is at least twice of those rotating number. On the other hand, if positive integers n and m are given and are satisfying $n > 2m$, then we can calculate the $\theta \in (0, 90)$ with $\theta = \frac{q}{p} = 90 - 180\frac{m}{n}$. Specifically, we note that the following

$$\phi : \theta \longmapsto \frac{90 - \theta}{180} = \frac{m}{n}$$

is a strictly decreasing function from those rationals of $(0, 90)$ to those of $\left(0, \frac{1}{2}\right)$. It is obvious to see that

$$\begin{aligned} \lim_{\theta \to 0^+} \frac{m}{n} &= \frac{1}{2}, \text{ and} \\ \lim_{\theta \to 90^-} \frac{m}{n} &= 0. \end{aligned} \tag{4}$$

In view of the first equation of (4), it is easy to prove that

$$\lim_{\theta\to 0^+} m = \infty \text{ and } \lim_{\theta\to 0^+} n = \infty. \tag{5}$$

Suppose $\lim_{\theta\to 0^+} n = \infty$ is false, then there exists positive integers N and m_k, n_k, where $k = 1, 2, ...,$ satisfying

$$\frac{m_1}{n_1} < \frac{m_2}{n_2} < ... < \frac{m_k}{n_k} < ... < \frac{1}{2}, \tag{6}$$

where

$$n_k \leq N, m_k < n_k/2 < N/2, \tag{7}$$

where $k = 1, 2, ...$. It follows from the equation of (6) that $\left\{\frac{m_k}{n_k}\right\}_{k=1}^{\infty}$ is an infinite set, and yet (7) suggests it is a finite set. Therefore, $\lim_{\theta\to 0^+} n = \infty$ and thus $\lim_{\theta\to 0^+} m = \lim_{\theta\to 0^+} \left(\frac{m}{n}\cdot n\right) = \infty$. Analogously, we can apply the second equation of (4) to show that

$$\lim_{\theta\to 90^-} n = \infty. \tag{8}$$

In summary, the equations of (4), (5) and (8) suggests the followings:

1. When the reflection rational angle θ is approaching 0 or 90 degrees, the periodic number of n is approaching infinity.

2. When the reflection rational angle θ is approaching 0 degrees, the rotating number m is approaching infinity too and m is generally half of n.

3. The reflections on circles form a convex polygon if and only if $m = 1$. In other words, the starting point A_0 goes around the circle only once.

4. Now if the reflection angle $\theta = \frac{q}{p}$ is a rational, we may ask when $m = 1$.In view of the equation of (3), $m = 1$ if and only if $180p$ is divisible by $90 - q$. For example, we can see from the following Table 1 that when $\theta = 45, 54, 70$ and 75 degrees, we see $180p$ is divisible by $90 - q$.

degree	gons	m/n	degree	gons	m/n	degree	gons	m/n
1	180	89/180	31	180	59/180	61	180	29/180
2	45	22/45	32	90	29/90	62	45	7/45
3	60	29/60	33	60	19/60	63	20	3/20
4	90	43/90	34	45	14/45	64	90	13/90
5	36	17/36	35	36	11/36	65	36	5/36
6	15	7/15	36	10	3/10	66	15	2/15
7	180	83/180	37	180	53/180	67	180	23/180
8	90	41/90	38	45	13/45	68	90	11/90
9	20	9/20	39	60	17/60	69	60	7/60
10	9	4/9	40	18	5/18	70	9	1/9
11	180	79/180	41	180	49/180	71	180	19/180
12	30	13/30	42	15	4/15	72	10	1/10
13	180	77/180	43	180	47/180	73	180	17/180
14	45	19/45	44	90	23/90	74	45	4/45
15	12	5/12	45	4	1/4	75	12	1/12
16	90	37/90	46	45	11/45	76	90	7/90
17	180	73/180	47	180	43/180	77	180	13/180
18	5	2/5	48	30	7/30	78	15	1/15
19	180	71/180	49	180	41/180	79	180	11/180
20	18	7/18	50	9	2/9	80	18	1/18
21	60	23/60	51	60	13/60	81	20	1/20
22	45	17/45	52	90	19/90	82	45	2/45
23	180	67/180	53	180	37/180	83	180	7/180
24	30	11/30	54	5	1/5	84	30	1/30
25	36	13/36	55	36	7/36	85	36	1/36
26	45	16/45	56	90	17/90	86	45	1/45
27	20	7/20	57	60	11/60	87	60	1/60
28	90	31/90	58	45	8/45	88	90	1/90
29	180	61/180	59	180	31/180	89	180	1/180
30	3	1/3	60	6	1/6	90	1	0

Table 1. When θ is an integer degree

The following is an immediate observation from the preceding result:

Corollary 10 *If the reflection angle θ is a rational number in degree(s), and we reduce $\frac{90-\theta}{180}$ to be the lowest term $\frac{m}{n}$. Then the denominator n is the minimum integer for the circle refections become periodic and the numerator m is the rounding number. On the other hand, if the reflection angle is an irrational in degree(s), then the reflections along a circle will not be periodic.*

Example 11 *Assume $\theta = \frac{q}{p} = \frac{7}{2}$, then we see that* $\gcd(90 \cdot 2 - 7, 180 \cdot 2) = 1$, *then the minimum number for the reflections become periodic is $n = 180p = 360$. The rotating number is $m = 90 \cdot 2 - 7 = 173$.*

Example 12 *Assume* $\theta = 31.1 = \frac{312}{10} = \frac{q}{p}$, *then we see* $\gcd(90p - q, 180p) = \gcd(90 \cdot 10 - 312, 180 \cdot 10) = 12$ *and hence* $n = \frac{180 \cdot 10}{12} = 150$ *and* $m = \frac{90 \cdot 10 - 312}{12} = 49$.

5 Replace Incoming And Outgoing Line Segments With Symmetric Curves

Mathematically, an incoming light and an outgoing light is symmetric to a normal line at a point on the circle. In other words, we may say that the outgoing line is the inverse of the incoming line with respect to the normal vector at a given point. Now, suppose we replace the incoming line by a smooth curve connecting pre-determined starting and terminating points, and we would like to find the inverse of this smooth curve with respect to a normal vector at a specified point over a circle. Since circles are symmetric, we expect to create nice patterns of graphs. According to [10], if $[p(t), q(t)]$ is the inverse of $[x(t), y(t)]$ with respective to the line $ax + by + d = 0$, where $t \in [t_1, t_2]$, then we have

$$\begin{aligned} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} &= \frac{1}{a^2+b^2} \begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix} \begin{bmatrix} x(t)-0 \\ y(t)-(\frac{-d}{b}) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d}{b} \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x(t)-0 \\ y(t)-(\frac{-d}{b}) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d}{b} \end{bmatrix}. \end{aligned} \tag{9}$$

We describe here how can replace the lines, represented by $[x(t), y(t)]$, by curves with respective to the proper normal vectors at appropriate points on the circle.

One easiest way to experiment this is drawing a regular convex polygon inscribed in a circle. The vertices of those regular convex polygons serve the following purposes:

1. The normal vectors at those vertices, $P_1, P_2, ..., P_n$, will serve as lines of symmetry when we apply the formula (9) in finding the general inverse of $[x_i(t_i), y_i(t_i)], i = 1, 2, ...n$, with respect to those lines of symmetry.

2. The respective vertices, $P_1, P_2, ..., P_n$, also represent proper starting point and end point for $[x_i(t_i), y_i(t_i)], i = 1, 2, ...n$.

The next Example 15 shows if we start with an equilateral inscribed in a circle, how we can apply our strategy to construct nice curves.

Example 13 *Consider the equilateral* $\triangle MNL$ *is an inscribed in the circle* $x^2 + y^2 = 4$, *where* $M = ([1.45596, 1.3712])$, $N = (0.45951, -1.9465)$ *and* $L = (-1.91547, 0.575301)$. *We describe how we can construct three ellipses that passes through* M, N *and* L *(shown in Figure 12). In other words, we replace the equilateral line segments inscribed in a circle by smooth curves when* *connecting two consecutive points on the equilateral.*

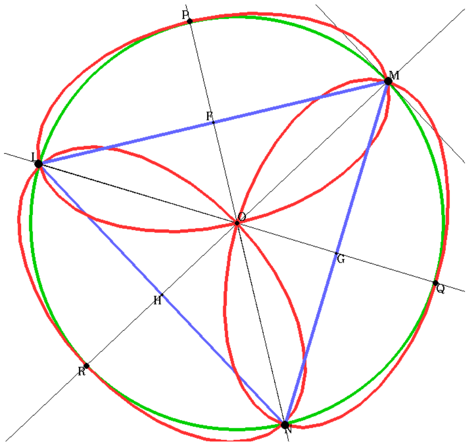

Figure 12. Replace an equilateral with smooth curves

We describe how we construct three ellipses that passes through M, N and L as follows:

Step 1. We find the midpoint F of ML and construct a perpendicular line l_1 to ML using F as the perpendicular foot. We call the intersection between l_1 and the circle to be P.

Step 2. We construct an ellipse using F as its center and FM and FP as its major and minor axes.

Step 3. We proceed to construct the second and third ellipses analogously.

Step 4. It is easy to see that the portion of the ellipse passing through L, P and M and the corresponding portion of the ellipse passing through M, Q and N are symmetric with respect to the line MR.

Step 5. Similarly, the portion of the ellipse passing through M, Q and N is symmetric to the portion of the ellipse passing through N, R and L are symmetric with respect to the line RH.

Remark: Incidentally, we came up with a construction of the rose with three leaves, where the angles between each leaf is $\frac{2\pi}{3}$.

In Example 16, we start with a square that is inscribed in a circle and we construct curves, which serve as general inverses with respect to proper normal vectors at respective four vertices.

Example 14 *Consider the square $ABCD$, where $A = \left(-\sqrt{2}, \sqrt{2}\right), B = \left(\sqrt{2}, \sqrt{2}\right), C = \left(\sqrt{2}, -\sqrt{2}\right)$ and $D = \left(-\sqrt{2}, -\sqrt{2}\right)$. In addition, E, I, G and K are midpoints of AB, BC, CD and DA respectively (see Figure 13). Construct symmetric curves which uses the normal vectors as the lines of symmetry at the respective vertices of the square $ABCD$ that is inscribed in a circle.*

We describe how we construct the symmetric curves as follows:

Step 1. We start with a square $ABCD$ that is inscribed in a given circle.

Step 2. We construct the perpendicular bisector of AB at E, where F is on the circle such that $EF \perp AB$.

Step 3. We construct the ellipse, which is centered at E and uses EA and EF as its major and minor axes respectively.

Step 4. We apply the same construction processes through the sides BC, CD and DA to obtain the Figure 13 below.

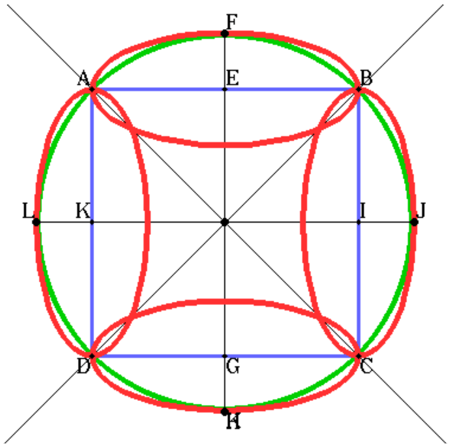

Figure 13. Replace a squre by smooth curves

Example 15 *Consider the preceding Example 16 with the circle of $x^2 + y^2 = 4$ and the squares $ABCD$. We describe here how how we can replace the straight edges AB, BC, CD, DA by smooth curves P_1, P_2, P_3 and P_4 respectively so that the following conditions are met: (a) P_1 and P_2 are symmetry with respective BD, (b) P_2 and P_3 are symmetric with respective CA, (c) P_3 and P_4 are symmetric with respective to DA.*

Step1. We refer to Figure 13. We let G be the midpoint of EF. We construct the parabola P_1 passing through A, G and B. We find $P_1(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t \\ \left(\frac{1}{4}\sqrt{2} - \frac{1}{2}\right) t^2 + \frac{2+\sqrt{2}}{2} \end{bmatrix}$. Note that the starting and terminating points for P_1 are at A and B respectively. Therefore, we choose $t \in \left[-\sqrt{2}, \sqrt{2}\right]$.

Step 2. To find P_2, we apply the general inverse $(p(t), q(t))$ for a parametric equation $(x(t), y(t))$ with respect to a line of $ax + by + d = 0$ (i.e., $y = \frac{-a}{b}x + \frac{-d}{b}$). In our case, $y = x, \theta = \frac{\pi}{4}$, $x - y = 0$, $a = 1, b = -1, d = 0$, so we write

$$P_2(t) = \begin{bmatrix} p_1(t) \\ q_1(t) \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{10}$$

$$= \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & -\cos\frac{\pi}{2} \end{bmatrix} \begin{bmatrix} t \\ \left(\frac{1}{4}\sqrt{2} - \frac{1}{2}\right) t^2 + \frac{2+\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{4}\sqrt{2} - \frac{1}{2}\right) t^2 + \frac{1}{2}\sqrt{2} + 1 \\ t \end{bmatrix}, \tag{11}$$

where $t \in \left[-\sqrt{2}, \sqrt{2}\right]$.

Step 3. Next, find the parabola P_3 that is symmetric to P_2 with respect to $y = -x$. We need to find the reflection of $\begin{bmatrix} p_1(t) \\ q_1(t) \end{bmatrix}$ with respect to $y = -x$ as follows, where $\theta = -\frac{\pi}{4}$. Thus,

$$\begin{aligned} P_3(t) = \begin{bmatrix} p_2(t) \\ q_2(t) \end{bmatrix} &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} p_1(t) \\ q_1(t) \end{bmatrix} \\ &= \begin{bmatrix} \cos \frac{-\pi}{2} & \sin \frac{-\pi}{2} \\ \sin \frac{-\pi}{2} & -\cos \frac{-\pi}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1}{4}\sqrt{2} - \frac{1}{2}\right) t^2 + \frac{1}{2}\sqrt{2} + 1 \\ t \end{bmatrix} \\ &= \begin{bmatrix} -t \\ \left(-\frac{1}{4}\sqrt{2} + \frac{1}{2}\right) t^2 - \frac{1}{2}\sqrt{2} - 1 \end{bmatrix}, \end{aligned}$$

where $t \in \left[-\sqrt{2}, \sqrt{2}\right]$.

Step 4. Finally, we find the reflection of $\begin{bmatrix} p_2(t) \\ q_2(t) \end{bmatrix}$ with respect to $y = x$ and $\theta = \frac{\pi}{4}$ as follows:

$$\begin{aligned} P_4(t) = \begin{bmatrix} p_3(t) \\ q_3(t) \end{bmatrix} &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} p_2(t) \\ q_2(t) \end{bmatrix} \\ &= \begin{bmatrix} \cos \frac{\pi}{2} & \sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & -\cos \frac{\pi}{2} \end{bmatrix} \begin{bmatrix} -t \\ \left(-\frac{1}{4}\sqrt{2} + \frac{1}{2}\right) t^2 - \frac{1}{2}\sqrt{2} - 1 \end{bmatrix} \\ &= \begin{bmatrix} \left(-\frac{1}{4}\sqrt{2} + \frac{1}{2}\right) t^2 - \frac{1}{2}\sqrt{2} - 1 \\ -t \end{bmatrix}, \end{aligned}$$

where $t \in \left[-\sqrt{2}, \sqrt{2}\right]$. We depict the plot as follows:

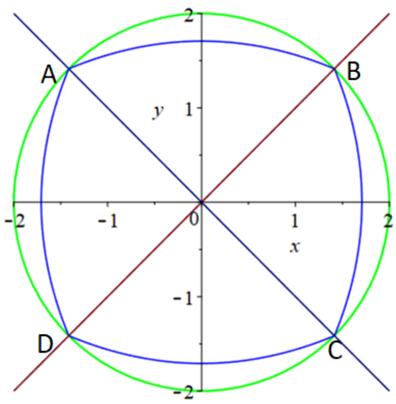

Figure 14. Reflections and parametric equations

6 Further Explorations

We certainly can extend the reflections over a circle to an ellipse as described in [4]. In which we describe if a trajectory closes after a finite number of bounces or the reflections along ellipses become periodic. The history of Poncelet's Theorem is very interesting and there are many deep mathematical results in connection with this theorem, including the conditions for periodicity obtained by Cayley. We refer readers to [4] in exploring several interesting scenarios regarding the elliptical billiards when technological tools are implemented. In what follows, we use a DGS and CAS [8], which developed by a Chinese research group, to explore the following three known facts which are proved by [9]. In the following demonstrations with technological tools, we shall see that even though the proofs in [9] are evidently non-trivial; however, technological tools can indeed be effectively implemented for making complex mathematical concepts more accessible.

Exploration 1. If the trajectory crosses the foci, then the reflected ray will pass the other foci. It can also be shown theoretically that the trajectory of the billiard converges to the major axis of the ellipse.

Example 16 *Consider the ellipse of* $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$ *with the foci of* $(-4,0)$ *and* $(4,0)$*. Let* P *be a point on the ellipse. We explore if the incidental ray* EP *passes one of the foci, then the reflected ray will pass the other foci. It can also be shown theoretically that the trajectory of the billiard converges to the major axis of the ellipse.*

Readers can explore this example through [S8] or *https://www.netpad.net.cn/svg.html#posts/137109.*

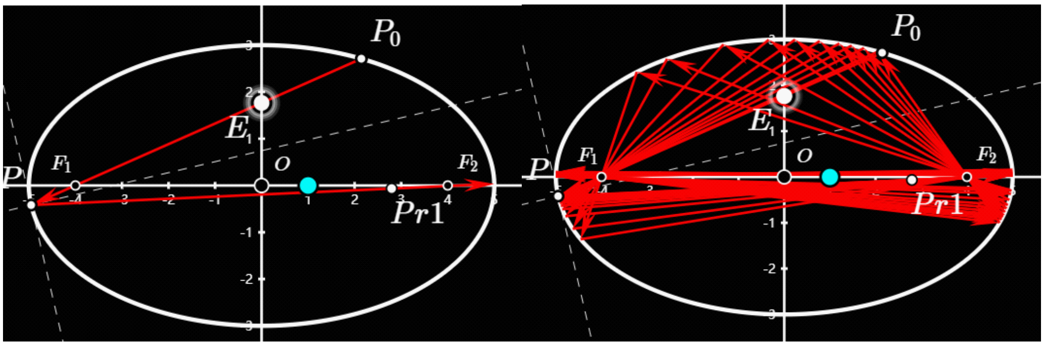

Figure 15(a). Reflections when incidental ray passes one foci.

Figure 15(b). Trajectory converges to the major axis of the ellipse.

Exploration 2. If the incidental ray EP crosses the line segment between the two foci, then we can show theoretically that the caustic forms a hyperbola.

Example 17 *Consider the ellipse of* $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$ *with the foci of* $(-4,0)$ *and* $(4,0)$*. Let* P *be a point on the ellipse. We explore if the incidental ray* EP *crosses the* $x-axis$ *between the two foci. Then we can show theoretically that the caustic forms a hyperbola.*

Readers can explore this example by modifying the example from [S9] or *https://www.netpad.net.cn/svg.*

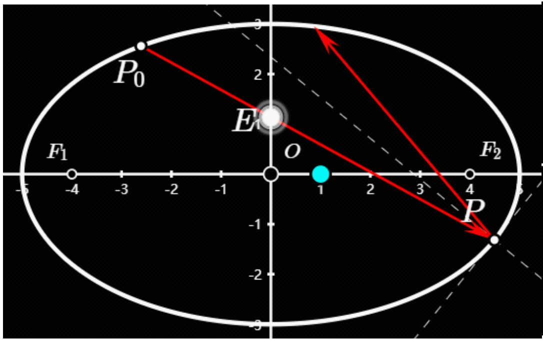

Figure 16(a). Incidental ray crosses between the two foci.

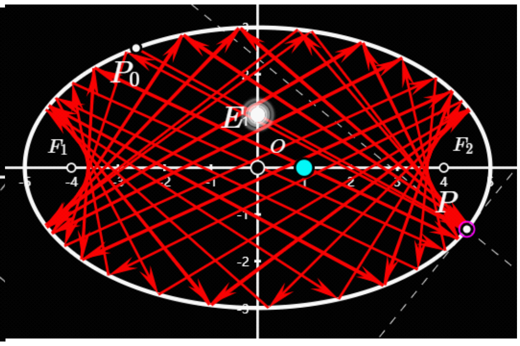

Figure 16(b). The caustic forms a hyperbola.

Exploration 3. If the incidental ray EP does not cross the line segment between the two foci, then it can be shown that every trajectory of the billiard is tangent to the ellipse which shares the same foci with the ellipse. In other words, the trajectory forms a caustic which is an ellipse confocal to the elliptical billiard table.

Example 18 *Consider the ellipse of* $\frac{x^2}{4.5^2} + \frac{y^2}{4.3^2} = 1$ *with their respective foci. Let* P *be a point on the ellipse. We explore if the incidental ray* EP *does not intersect with the line segment between the two foci of the ellipse. Then every trajectory of the billiard is tangent to the ellipse which shares the same foci with the ellipse. In other words, the trajectory has a caustic which is an ellipse confocal to the elliptical billiard table.*

Readers can explore this example by modifying the example from [S10] or *https://www.netpad.net.cn/svg*

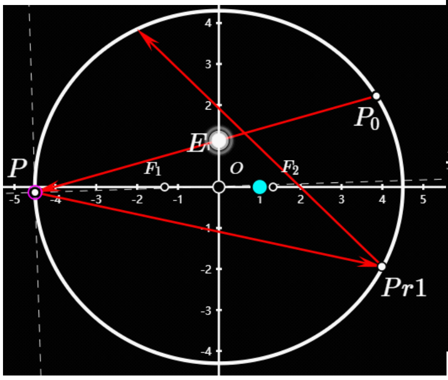

Figure 17(a) Incidental ray does not intersect the line segment containing two foci.

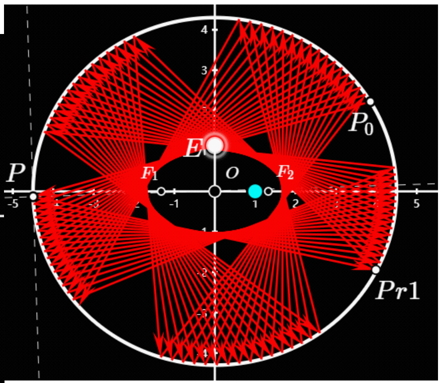

Figure 17(b). A caustic confocal to the elliptical billiard table

Remark: By looking at the Figure 17(b) alone, although the ellipse is close to a circle, one can firmly detect that the reflections are not along a circle since its caustic forms an ellipse instead of a circle.

7 Conclusions

Typically students are allocated no more than 10 minutes to solve one problem in a Gaokao (College Entrance Exam) in China. Under such circumstances, it is not hard to imagine that many students may decide to give up to solving challenging problems. It is clear that technological tools can provide us with crucial intuition before we attempt more rigorous analytical solutions. Here we have gained geometric intuitions while using a DGS. In the meantime, we use the computer algebra system (CAS) for verifying that our analytical solutions are consistent with our initial intuitions. In this paper, we started with a simple reflection problem from Gaokao and investigated several scenarios using technological tools. The complexity level of the problems we posed vary from the simple to the difficult. With the interactive activities presented at the Supplementary Electronic Materials, we have made learning mathematics to be fun, accessible and yet challenging. Activities presented in this paper definitely are accessible to those teachers' training programs.

We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate problems ranging from the simple to the more challenging. We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important task for many educators and researchers.

8 Acknowledgements

The author would like to thank the research group of Chinese Academic of Sciences in Chengdu for providing inspiring constructions of dynamic geometry worksheets for this paper.

9 Supplementary Electronic Materials

[S1] Netpad worksheet for Section 2.1 number 1: https://www.netpad.net.cn/svg.html#posts/152541.

[S2] Netpad worksheet for Section 2.1 number 2: https://www.netpad.net.cn/svg.html#posts/153482.

[S3] Netpad worksheet for Section 2.1 number 3: https://www.netpad.net.cn/svg.html#posts/142435.

[S4] GeoGebra worksheet for Section 2.1 number 4.

[S5] GeoGebra worksheet for $\theta = 15°$.

[S6] Maple worksheet for $\theta = 15°$.

[S7] Netpad worksheet for $\theta = 15°$: *https://www.netpad.net.cn/svg.html#posts/159437.*

[S8] Netpad worksheet for Example 16: *https://www.netpad.net.cn/svg.html#posts/137109.*

[S9] Netpad worksheet for Example 17: *https://www.netpad.net.cn/svg.html#posts/158728.*

[S10] Netpad worksheet for Example 18: *https://www.netpad.net.cn/svg.html#posts/137109.*

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